SOME TOPICS FROM RINGS AND MODULES

Carol L. Walker and Elbert A. Walker

Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico

January 2010

Contents

Preface

This is an outline for a course in classical rings and modules.

Introduction

Definition 0.0.1 *A **binary operation** on a set S is a function $S \times S \longrightarrow S$.*

The image of $(a, b) \in S \times S$ is often written as $a \circ b$, $a \cdot b$ or ab (and called *multiplication*) or as $a + b$ (and called *addition*).

Definition 0.0.2 *A **group** $(G, \circ)$ is a nonempty set G together with a binary operation $\circ$ on G such that*

1. *$a \circ (b \circ c) = (a \circ b) \circ c$ for all a, b, $c \in G$ (associativity),*
2. *there exists $e \in G$ such that $a \circ e = e \circ a = a$ for all $a \in G$ (identity),and*
3. *for each $a \in G$ there exists $a' \in G$ such that $a \circ a' = a' \circ a = e$ (inverse).*

The element e is called an **identity**. The identity is usually written as 1 if the operation is multiplication and as 0 if the operation is addition. The element a' is called the **inverse** of a is written as a^{-1} (multiplication) or as $-a$ (addition).

Definition 0.0.3 *A group $(G, \circ)$ is called **commutative** or **abelian** if $a \circ b = b \circ a$ for all a, $b \in G$.*

Definition 0.0.4 *A **ring** $(R, +, \circ)$ is a nonempty set R together with two binary operations $+$ and $\circ$ on R such that*

1. *$(R, +)$ is a commutative group,*
2. *$a \circ (b \circ c) = (a \circ b) \circ c$ for all a, b, $c \in R$ (associativity),*
3. *$a \circ (b + c) = a \circ b + a \circ c$ and $(a + b) \circ c = a \circ c + b \circ c$ for all a, b, $c \in R$ (distributive properties).*

Definition 0.0.5 *A ring $(R, +, \circ)$ is **commutative** if $a \circ b = b \circ a$ for all a, $b \in R$.*

Definition 0.0.6 *A ring $(R, +, \circ)$ is a **ring with identity** if there exists an element $e \in R$ such that $a \circ e = e \circ a = a$ for all $a \in R$.*

Definition 0.0.7 *A* ***left (right) ideal*** I *of a ring* R *is non-empty subset of* R *that is itself a ring under the operations in* R *such that* $ra \in I$ $(ar \in I)$ *for all* $r \in R$ *and* $a \in i$*. An* ***ideal*** *is both a left and a right ideal.*

Definition 0.0.8 *If* I *is an ideal of a ring* R*, the* ***quotient ring*** R *modulo* I *is the set of cosets* $R/I = \{a + I : a \in R\}$ *where addition and multiplication are given by* $(a + I) + (b + I) = a + b + I$ *and* $(a + I)(b + I) = ab + I$*.*

Definition 0.0.9 *A* ***left*** R***-module*** *is a commutative group* $(M, +)$ *together with a function* $\circ : R \times M \longrightarrow M$ *such that*

1. $a \circ (m + n) = a \circ m + a \circ n$ *for all* $a \in R$ *and for all* m, $n \in M$,
2. $(a + b) \circ m = a \circ m + b \circ n$ *for all* a, $b \in R$ *and for all* $m \in M$*, and*
3. $(ab) \circ m = a \circ (b \circ m)$ *for all* a, $b \in R$ *and for all* $m \in M$*.*

In what follows, all R-modules will be left R-modules.

Definition 0.0.10 *An* R*-module* M *is* ***unitary*** *if* R *is a ring with identity* 1 *and* $1 \circ m = m$ *for all* $m \in M$*.*

In what follows, multiplication will be indicated by juxtaposition and $+$ will be used for both ring addition and addition in the group $(M, +)$.

Definition 0.0.11 *A* ***submodule*** N *of an* R*-module* M *is a subset of* M *which is itself an* R*-module with respect to addition and scalar multiplication in* M*.*

Definition 0.0.12 *A collection* Ω *of submodules of an* R*-module* M *is a* ***chain*** *if for any pair* S *and* T *in* Ω*, elther* $S \subseteq T$ *or* $T \subseteq S$*.*

Definition 0.0.13 *If* M *and* N *are* R*-modules, then a function* $f : M \longrightarrow N$ *is an* R***-module homomorphism*** *if* $f(m + n) = f(m) + f(n)$ *and* $f(am) = af(m)$ *for all* m, $n \in M$ *and* $a \in R$*.*

Definition 0.0.14 *Given an* R*-module homomorphism* $f : M \longrightarrow N$*, the* ***kernel*** *of* f *is* $\ker(f) = \{m \in M : f(m) = 0\}$ *and* ***image*** *of* f *is* $\operatorname{Im}(f) = \{f(m) : m \in M\}$*.*

Definition 0.0.15 *A sequence* $M \xrightarrow{f} N \xrightarrow{g} P$ *of* R*-module homomorphisms is called* ***exact*** *at* N *if* $\operatorname{Im}(f) = \ker(g)$*.*

Definition 0.0.16 *A sequence* $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ *of of* R*-module homomorphisms is called a* ***short exact sequence*** *of* R*-module homomorphisms if it is exact at each point in the sequence. (The homomorphisms on the extreme left and right are the maps that take everything to* 0*.)*

Definition 0.0.17 *An* R*-module* M *is* ***simple*** *if* M *is non-zero and has no proper non-zero submodules.*

Chapter 1

Some Classical Theorems

1.1 Chain Conditions

Rings will be rings with identity, and modules will be unitary left modules unless otherwise stated.

Definition 1.1.1 *An R-module M satisfies the **ascending chain condition** if every ascending chain of submodules of M has only finitely many terms. The module M satisfies the **maximum condition** if every non-empty set of submodules of M has a maximal element.*

Theorem 1.1.2 *Let R be a ring. The following are equivalent.*

1. *M satisfies the ascending chain condition.*
2. *M satisfies the maximum condition.*
3. *Every submodule of M is finitely generated.*

Definition 1.1.3 *An R-module M is **Noetherian** if M satisfies the ascending chain condition. The ring R is **left Noetherian** if it is Noetherian as a left module over itself.*

Definition 1.1.4 *An R-module M satisfies the **descending chain condition** if every descending chain of submodules of M has only finitely many terms. The module M satisfies the **minimum condition** if every non-empty set of submodules of M has a minimal element.*

Theorem 1.1.5 *An R-module M satisfies the descending chain condition if and only if M satisfies the minimum condition.*

Definition 1.1.6 *An R-module M is **Artinian** if it satisfies the descending chain condition. The ring R is **left Artinian** if R is Artinian as a left module over itself.*

Theorem 1.1.7 *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then B is Noetherian (Artinian) if and only if A and C are Noetherian (Artinian).*

Corollary 1.1.8 *A ring is left Noetherian (Artinian) if and only if every finitely generated left R-module is Noetherian (Artinian).*

Definition 1.1.9 *A finite descending chain*

$$M = M_0 \supset M_1 \supset ... \supset M_r = 0$$

*of submodules of M is a **Jordan-Hölder series** if all the factors M_i/M_{i+1} are simple modules. Two such series are **equivalent** if there is a one-to-one correspondence between the factors such that corresponding factors are isomorphic.*

Theorem 1.1.10 *(Jordan-Hölder) Any two Jordan-Hölder series of a module are equivalent.*

Theorem 1.1.11 *An R-module satisfies both the ascending and the descending chain conditions if and only if it has a Jordan-Hölder series.*

1.2 The Radical of a Ring

Definition 1.2.1 *An R-module is **faithful** if the annihilator $0 : M = \{r \in R : rM = 0$ of M is 0. A ring R is **left primitive** if there exists a faithful simple left R-module. An ideal $I \neq R$ is **left primitive** if the ring R/I is left primitive.*

Remark 1.2.2 *A primitive ring R has a faithful simple module S, $E = Hom_R(S, S)$ is a division ring, and multiplication by elements of R are E-homomorphisms of S and thus linear transformations on the vector space S. Thus R is a ring of linear transformations on S, and S has no invariant subspaces with respect to this ring R.*

Theorem 1.2.3 *Every maximal left ideal contains a left primitive ideal. Every left primitive ideal is the intersection of all maximal left ideals containing it.*

Corollary 1.2.4 *The intersection of all maximal left ideals of a ring is the intersection of all left primitive ideals of that ring and is a two-sided ideal.*

Theorem 1.2.5 *Let I be a left ideal of R. The following are equivalent.*

1. *$1 + a$ has a left inverse for all $a \in I$.*

2. *I is contained in every maximal left ideal.*

Lemma 1.2.6 *If I is a left ideal and $1 + a$ has a left inverse for all a in I, then $1 + a$ has a right inverse for all a in I.*

Theorem 1.2.7 *The intersection of all left ideals, the intersection of all right ideals, and intersection of all left primitive ideals, and the intersection of all right primitive ideals are the same.*

Definition 1.2.8 *The intersection of all maximal left ideals of a ring R is the* ***Jacobson radical*** *of R, denoted $rad(R)$ or $J(R)$.*

Theorem 1.2.9 *The Jacobson radical satisfies $rad(R) = \{a \in R : 1 + ra$ has a left inverse for all $r \in R\} = \{a \in R : 1 + ra$ has a right inverse for all $r \in R\}$.*

Definition 1.2.10 *An element r in R is* ***nilpotent*** *if $r^n = 0$ for some positive integer n. A left (or right) ideal I is* ***nil*** *if every element of it is nilpotent. A left (or right) ideal is* ***nilpotent*** *if $I^n = 0$ for some positive integer n.*

Theorem 1.2.11 *The Jacobson radical $rad(R)$ contains all nil left and all nil right ideals.*

Lemma 1.2.12 *(Nakayama) The following are equivalent for the left ideal I of a ring R.*

1. *$I \subset rad(R)$.*
2. *If M is a finitely generated left R-module and $IM = 0$, then $M = 0$.*

Corollary 1.2.13 *If R is left Artinian, then $rad(R)$ is nilpotent.*

Lemma 1.2.14 *Let R be a ring. The following hold.*

1. *The sum of a finite number of nilpotent left ideals of R is nilpotent.*
2. *The sum of any set of nil ideals of R is nil.*
3. *The sum of all the nilpotent ideals of R is nil and contains all nilpotent one-sided ideals.*

Corollary 1.2.15 *If R is left Artinian, then $rad(R)$, the sum of all nil left ideals, the sum of all nil right ideals, the sum of all nilpotent left ideals, and the sum of all nilpotent right ideals are the same.*

Corollary 1.2.16 *If R is left Noetherian, then R has a maximum nilpotent left ideal and a maximum nilpotent ideal.*

Lemma 1.2.17 *A ring R contains no non-zero nil left ideals if and only if R contains no non-zero nil right ideals.*

Definition 1.2.18 *A left ideal I is an* ***annihilator left ideal*** *if it is the left annihilator of a subset of R.*

Lemma 1.2.19 *Let R satisfy the maximum condition on annihilator left ideals. If R has a non-zero nil right or left ideal, then R has a non-zero nilpotent ideal.*

Theorem 1.2.20 *(Levitzki) If R is left Noetherian, then every nil left or nil right ideal is nilpotent.*

1.3 Semi-simple Rings

Definition 1.3.1 *An R-module is **semi-simple** if it is the direct sum of simple R-modules. The ring R is **left semi-simple** if R is semi-simple as a left module over itself.*

Theorem 1.3.2 *The module M is semi-simple if and only if every submodule of M is a direst summand of M.*

Theorem 1.3.3 *If $0 \to L \to M \to N \to 0$ is exact with M semi-simple, then L and N are semi-simple.*

Theorem 1.3.4 *Every left R-module is semi-simple if and only if R is left semi-simple.*

Lemma 1.3.5 *A left ideal I of R is a summand of the left R-module R if and only if $I = Re$ for some idempotent e of R.*

Lemma 1.3.6 *A minimal left ideal I of R is a summand of R if and only if $I^2 \neq 0$.*

Theorem 1.3.7 *The following are equivalent.*

1. *R is left semi-simple.*
2. *R is left Artinian with $rad(R) = 0$.*
3. *R is left Artinian, and for I a minimal left ideal, $I^2 \neq 0$.*

Theorem 1.3.8 *(Levitzi) If R is left Artinian, then R is left Noetherian.*

Theorem 1.3.9 *A left Artinian ring R is simple if and only if there is a simple faithful R-module. If R is simple and left Artinian, then all simple left $rM-$ mocules are isomorphic.*

Corollary 1.3.10 *If R is left Artinian, then a two-sided ideal I is left primitive if and only if it is maximal.*

Corollary 1.3.11 *If R is left Artinian, then $rad(R)$ is the intersection of all maximal two-sided ideals.*

Theorem 1.3.12 *If R is left semi-simple, then there are only a finite number of isomorphic classes of simple left R-modules. The correspondence $M \longleftrightarrow 0 : M$ is bijective between isomorphism simple R-modules and maximal two-sided ideals of R.*

Theorem 1.3.13 *A left semi-simple ring R is the direct sum of its minimal two-sided ideals.*

Corollary 1.3.14 *A left semi-simple ring R is uniquely the direct sum of simple left Artinian rings.*

Corollary 1.3.15 *Let $M_1, M_2, ..., M_n$ be the maximal two-sided ideals in a left semi-simple ring R. Then*

$$R \cong R/M_1 \oplus R/M_2 \oplus ... \oplus R/M_n$$

1.4 The Jacobson Density Theorem

Theorem 1.4.1 *(The Jacobson Density Theorem) Let R be a left primitive ring, S a simple faithful left R-module, and $E = Hom_R(S, S)$. Then S is a vector space over the division ring E, and if $\{s_1, s_2, ..., s_n\}$ is linearly independent in S and $\{x_1, x_2, ..., x_n\} \subset S$, then there is an element r in R such that $rs_i = x_i$ for all i.*

Corollary 1.4.2 *If the dimension of S is n, then R is isomorphic to the ring of $n \times n$ matrices E_n over the division ring E.*

Corollary 1.4.3 *If R is left primitive and left Artinian, then the dimension of S is finite, and so $R \cong E_n$.*

Corollary 1.4.4 *(Wedderburn) If R is simple and left Artinian, then $R \cong E_n$.*

Theorem 1.4.5 *Let D be a division ring, and let V be a left vector space over D of finite dimension. Then the ring $E = Hom_D(V, V) = D_n$ is a simple ring which is left and right Artinian. The length of any Jordan-Hölder series for E as a left E-module is the same as the length of any such series for E as a right E-module, and is the dimension of V.*

Corollary 1.4.6 *A simple ring is left Artinian if and only if it is right Artinian.*

Theorem 1.4.7 *If D and E are division rings, m and n are positive integers, and $D_m \cong E_n$, then $D \cong E$ and $m = n$.*

Theorem 1.4.8 *A necessary and sufficient condition that a ring be semi-simple is that it be the direct sum of rings each of which is isomorphic to D_n for various division rings D and various positive integers n. Such a representation is unique. A ring is left semi-simple if it is right semi-simple.*

1.5 Exercises

1. Find an example of a ring that is left primitive but not right primitive. (G. Bergman, Proc. Amer. Math. Soc. 15(1964), 473,475.)
2. Prove that every left primitive ideal is contained in a right primitive ideal.
3. Prove that if R is left Noetherian and satisfies the descending chain condition on principal left (right) ideals, then R is left Artinian.
4. Find the radical of R_n in terms of the radical of t.
5. Find the radical of Z/Zm.
6. Prove that the radical of a ring contains no non-zero idempotents.
7. Let e be an idempotent in R. Prove that $rad(eRe) = e(rad(R))e$.

8. Let I be a two sided ideal contained in $rad(R)$. show that $rad(R/I) = rad(R)/I$. In particular, $rad(R/rad(R))) = 0$.

9. Prove that $rad(R) = 0$ if and only if there exists a faithful semi-simple left R-module.

10. Prove that if $rad(R) = 0$ and R has the descending chain condition on principle left ideals, then R is semi-simple.

11. Prove that R is left primitive if and only if R has a maximal left ideal which contains no non-zero two sided ideals.

12. Let e be idempotent in R. Prove that $Hom_R(Re, Re)$ is isomorphic to eRe.

13. Let e_1 and e_2 be idempotents in r. Prove that the left modules Re_1 and Re_2 are isomorphic if and only if the right modules e_1R and e_2R are isomorphic.

14. Let I be a minimal left ideal of R, and let S be the left ideal generated by all minimal left ideals isomorphic to I. Prove that S is two sided, and is a direct sum of left ideals isomorphic to I.

15. Prove that a simple ring with a minimal left ideal is left semi-simple.

16. Let D be a division ring, U an infinite dimensional vector space over E, and $R = Hom_D(U, U)$. Show that R has a maximal left ideal which contains no maximal two-sided ideal. Show that a primitive ring is not necessarily simple. Show that a left primitive ideal is not necessarily two sided.

17. Show that if $R_1, R_2, ..., R_n$ are left Artinian (Noetherian), then so is the ring direct sum $R_1 \oplus R_2 \oplus ... \oplus R_n$. What is the radical of this direct sum?

18. Prove that $(R/I)_n \cong R_n/I_n$ if I is any ideal of R.

19. Find all simple commutative rings.

20. Show that the radical of a commutative Artinian ring is the set of all nilpotent elements.

21. Let R be a commutative ring, and let N be the largest nil ideal of R. Find the largest nil ideal of $R[x]$.

22. Find the center of R_n.

23. Show that $R_n[x] \cong (R[x])_n$.

24. Prove that if R has no non-zero nilpotent elements, then any idempotent in R is in the center of R.

25. Prove that the left socle equals the right socle if R has no non-zero nilpotent ideals.

26. Prove that if R is isomorphic to a dense ring of linear transformations over a left vector space, then R is left primitive.

27. Let R be left Artinian, J its radical, and M a left R-module. Prove that if $JM = 0$, then M is semi-simple and $Hom_R(X, M) \cong Hom_{R/J}(X/JX, M)$.

28. Prove that if R is left Artinian, then there are only finitely many isomorphism classes of simple left R-modules.

29. A ring is **semi-primary** if $R/rad(R)$ is semi-simple and $rad(R)$ is nilpotent. Prove that if a semi-primary ring is left Noetherian, then it is left Artinian.

30. Let V be a vector space over D, $V \neq 0. =$, and let $E = Hom_D((V, V)$. Let c be the smallest infinite cardinal bigger than $\dim(V)$. Prove that there exists a one-to-one correspondence between the non-zero ideals of E and the set of infinite cardinals $\leq c$.

31. Let $E = Hom_D((V, V)$, D a division ring. Prove that there exists a one-to-one correspondence between ultra-filters of subspaces of V and the maximal left ideals of E. Find all maximal right ideals of E.

32. Show that the ring of all 2×2 matrices (a_{ij}) with a_{11} an integer, a_{12} and a_{22} rationals, and $a_{21} = 0$ is right Noetherian but not left Noetherian.

33. Show that the ring of all 2×2 matrices (a_{ij}) with a_{11} rational, a_{12} and a_{22} reals, and $a_{21} = 0$ is right Artinian but not left Artinian.

Chapter 2

Projective and Injective Modules

2.1 Projective Modules

Definition 2.1.1 *A module P is **projective** if every diagram*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact row can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Theorem 2.1.2 *Free modules are projective.*

Theorem 2.1.3 *A direct sum $\bigoplus_{i \in I} P_i$ is projective if and only if each P_i is projective.*

Theorem 2.1.4 *The following are equivalent.*

1. *P is projective.*

2. *P is a summand of a free module.*

3. *Every exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ splits.*

4. *If the sequence $B \longrightarrow C \longrightarrow 0$ is exact, the so is the sequence $Hom(P, B) \longrightarrow Hom(P, C) \longrightarrow 0$.*

Theorem 2.1.5 *(The dual basis lemma) P is projective if and only if there exist families $\{x_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ with $x_i \in P$ and $f_i \in Hom(P, R)$ such that for $x \in P$, $f_i(x) = 0$ almost always and $x = \sum_{i\in I} f_i(x)x_i$.*

Corollary 2.1.6 *P is finitely generated and projective if and only if there exist such families $\{x_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ with I finite.*

2.2 Injective Modules

Definition 2.2.1 *A module is **injective** if every diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & & \\ & & Q & & \end{array}$$

with exact row can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \swarrow & \\ & & Q & & \end{array}$$

Theorem 2.2.2 *A direct product $\prod_{i\in I} Q_i$ is injective if and only if every Q_i is injective.*

Theorem 2.2.3 *The left R-module Q is injective if and only if for each left ideal I of R, the diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & R \\ & & \downarrow & & \\ & & Q & & \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & R \\ & & \downarrow & \swarrow & \\ & & Q & & \end{array}$$

Corollary 2.2.4 *Let R be a left principal ideal ring with no zero divisors. A left R-module Q is injective if and only if $rQ = Q$ for every non-zero $r \in R$.*

Corollary 2.2.5 *An Abelian group is injective if and only if it is divisible.*

Theorem 2.2.6 *Every module is contained in an injective module.*

Theorem 2.2.7 *The following are equivalent.*

1. Q is injective.

2. Every exact sequence $0 \longrightarrow Q \longrightarrow N$ splits.

3. If the sequence $0 \longrightarrow A \longrightarrow B$ is exact, then so is the sequence $Hom(B, Q) \longrightarrow Hom(A, Q) \longrightarrow 0$.

Definition 2.2.8 *A submodule N of M is an* ***essential submodule*** *and M is an* ***essential extension*** *of N, if for every submodule S of M, $S \cap N = 0$ only if $S = 0$.*

Definition 2.2.9 *The module Q is an* ***injective envelope*** *of M is M is a submodule of Q, Q is injective, and no proper submodule of Q containing M is injective.*

Theorem 2.2.10 *If M is a submodule of the injective module Q, then any essential extension of M in Q is an injective envelope of M.*

Corollary 2.2.11 *The module Q is an injective envelope of M if and only if Q is injective and M is essential in Q.*

Theorem 2.2.12 *Every module has an injective envelope. If Q_1 and Q_2 are injective envelopes of M, then there is an isomorphism $Q_1 \longrightarrow Q_2$ fixing M elementwise.*

Theorem 2.2.13 *The module P is projective if and only if every diagram*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ Q_1 & \longrightarrow & Q_2 & \longrightarrow & 0 \end{array}$$

with exact row and with Q_1 injective can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ Q_1 & \longrightarrow & Q_2 & \longrightarrow & 0 \end{array}$$

Theorem 2.2.14 *The module Q is injective if and only if every diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_2 \\ & & \downarrow & & \\ & & Q & & \end{array}$$

with exact row and with P projective can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_2 \\ & & \downarrow & \swarrow & \\ & & Q & & \end{array}$$

2.3 Hereditary Rings

Definition 2.3.1 *The ring R is **left hereditary** if every left ideal of R is projective.*

Theorem 2.3.2 *If R is left hereditary, then every submodule of a free left R-module is a direct sum of modules each of which is isomorphic to a left ideal of R.*

Theorem 2.3.3 *The following are equivalent.*

1. *R is left hereditary.*
2. *Each submodule of a projective left R-module is projective.*
3. *Each homomorphic image of an injective left R-module is injective.*

Chapter 3

Direct Sum Decompositions

3.1 Azumaya Theorems

Definition 3.1.1 *A ring R is **local** if the non-units of R form an ideal. (Thus a local ring has a unique maximal left, right, and two-sided ideal.)*

Theorem 3.1.2 *(Azumaya) Suppose that*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{i \in J} N_j$$

with $Hom(N_j, N_j)$ local, with each M_i indecomposable, and with no M_i or N_i zero. Then there is a one-to-one correspondence $f : I \to J$ such that $M_i \cong N_{f(i)}$.

Theorem 3.1.3 *(Crawley-Jonsson-Warfield) Suppose that*

$$M = \bigoplus_{i \in I} M_i$$

with each M_i countably generated with local endomorphism ring, then any summand of M is a direct sum of such modules. In particular, any two direct sum decompositions of M refine to equivalent ones.

Theorem 3.1.4 *Let S be a summand of*

$$\oplus_{i=1}^{n} M_i$$

with each $Hom_R(M_i, M_i)$ local. Then S is a direct sum of a finite number of modules each isomorphic to some M_i.

Theorem 3.1.5 *Let*

$$\oplus_{i=1}^{n} M_i = \oplus_{i \in J} S_j$$

with each $Hom_R(M_i, M_i)$ local. Then the decomposition $\oplus_{i \in J} S_j$ refines to one equivalent to $\oplus_{i=1}^{n} M_i$.

Some more Azumaya theorems here, countable, finite decompositions, etc.

3.2 Some Direct Sum Decompositions for Projective Modules

Theorem 3.2.1 *(Kaplansky) Any direct summand of a direct sum of countably generated modules is a direct sum of countably generated modules.*

Theorem 3.2.2 *(C. Walker) Let m be an infinite cardinal. Any direct summand of a direct sum of modules each generated by a subset of cardinality $\leq m$ is a direct sum of modules each generated by a subset of cardinal $\leq m$.*

Theorem 3.2.3 *(C. Walker) Let m be an infinite cardinal. Any direct summand of a direct sum of modules of cardinality $\leq m$ is a direct sum of modules of cardinality $\leq m$.*

Theorem 3.2.4 *(Kaplansky) Projective modules are direct sums of countably generated modules.*

Lemma 3.2.5 *(Kaplansky) Let M be a countably generated module. Assume that any summand N of M has the property that any element of N can be embedded in a free (respectively, finitely generated) direct summand of N. Then M is free (respectively, a direct sum of finitely generated modules).*

Lemma 3.2.6 *(Kaplansky) Let P be a projective module over a local ring. Then any element of P can be embedded in a free summand of P.*

Theorem 3.2.7 *(Kaplansky) Any projective module over a local ring is free.*

Definition 3.2.8 *A ring R is **left semi-hereditary** if every finitely generated left ideal of R is projective.*

Lemma 3.2.9 *If R is left semi-hereditary, then every finitely generated submodule of a free left R-module is a direct sum of modules each of which is isomorphic to finitely generated left ideal of R. In particular, finitely generated submodules of projectives are projective.*

Lemma 3.2.10 *Let P be projective over a commutative semi-hereditary ring R. Then any element of P can be embedded in a finitely generated summand of P.*

Theorem 3.2.11 *(Kaplansky) Let R be a commutative semi-hereditary ring. Then any projective R-module is a direct sum of modules each of which is isomorphic to a finitely generated ideal of R.*

Corollary 3.2.12 *If R is a commutative integral domain such that every finitely generated ideal is principal, then every projective R-module is free.*

Definition 3.2.13 *A ring R is **regular** if for each $r \in R$, there is an $s \in R$ such that $rsr = r$.*

Lemma 3.2.14 *Finitely generated left ideals of regular rings are summands.*

Theorem 3.2.15 *A ring R is regular if and only if every finitely generated submodule of a projective left R-module is a summand.*

Corollary 3.2.16 *If R is regular, then finitely generated projective modules are isomorphic to direct sums of left ideals of R.*

Corollary 3.2.17 *If R is regular, then so is R_n.*

3.3 Some Direct Sum Decompositions for Injective Modules

Theorem 3.3.1 *A ring R is left Noetherian if and only if the direct sum of injective left R-modules is injective.*

Lemma 3.3.2 *Let Q be injective. The following are equivalent.*

1. *Q is the injective envelope of every one of its non-zero submodules.*
2. *Q contains no non-zero submodules S and T such that $S \cap T = 0$.*
3. *Q is indecomposable.*

Definition 3.3.3 *Let I be a left ideal of R, and suppose that*

$$I = I_1 \cap I_2 \cap \cdots \cap I_n$$

with the I_j left ideals. This is an ***irredundant decomposition*** *of I if no I_j contains the intersection of the rest.*

Theorem 3.3.4 *Let $I = I_1 \cap I_2 \cap ... \cap I_n$ be an irredundant decomposition of the left ideal I. Suppose that the injective envelope $E(R/I_j)$ is indecomposable. Then the natural embedding of R/I into $\oplus_j E(R/I_j)$ can be extended to $E(R/I)$, and this extension is an isomorphism.*

Definition 3.3.5 *A left ideal I of R is* ***irreducible*** *if $I = K \cap L$ implies that $I = K$ or $I = L$.*

Theorem 3.3.6 *The module Q is an indecomposable injective module if and only if $Q = E(R/I)$ with I irreducible. In this case, for x in Q, $x \neq 0$, $0 : x$ is irreducible and $Q = E(R/(0 : x))$.*

Lemma 3.3.7 *If I is a left ideal in the left Noetherian ring R, then I is the intersection of finitely many irreducible left ideals.*

Lemma 3.3.8 *The endomorphism ring of an indecomposable injective module is local.*

Theorem 3.3.9 *Let R be left Noetherian. Then any injective left R-module is a direct sum of indecomposable injective modules. Any two such decompositions are equivalent.*

Theorem 3.3.10 *Let S be a summand of $\oplus_{i=1}^{n} Q_i$ with the endomorphism of each Q_i being local. Then S is a direct sum of a finite number of modules each isomorphic to some Q_i.*

Corollary 3.3.11 *Let $\oplus_{i \in I} S_i = \oplus_{j \in J} Q_j$ with the endomorphism ring of each Q_j local Then the decomposition $\oplus_{i \in I} S_i$ refines to one equivalent to $\oplus_{j \in J} Q_j$.*

Theorem 3.3.12 *(Faith and E. Walker) Let $Q = \oplus_{i \in I} Q_i$ with Q_i countable generated, indecomposable and injective. Then any summand of Q is a direct sum of countable generated indecomposable modules.*

Theorem 3.3.13 *If each Q_i is indecomposable and injective, then any summand of $\oplus_{i=1}^{\infty} Q_i$ is a direct sum of modules each of which isomorphic to some Q_i.*

Corollary 3.3.14 *Let*

$$\bigoplus_{i=1}^{\infty} S_i = \bigoplus_{j=1}^{\infty} Q_j$$

with each Q_j indecomposable and injective. Then the decomposition on the left can be refined to one equivalent to the one on the right.

Theorem 3.3.15 *(Faith and E. Walker) A ring R is left Noetherian if and only if there exists a cardinal c such that each injective left R-module is a direct sum of modules each generated by c elements.*

Corollary 3.3.16 *(Papp) If each injective left R-module is a direct sum of indecomposable injective modules, then R is left Noetherian.*

Theorem 3.3.17 *A ring R is left Artinian if and only if every injective left R-module is the direct sum of injective envelopes of simple modules.*

3.4 Exercises

1. Let P be projective. Prove that there exists a free module F such that $P \oplus F \cong F$.

2. If $P \to M \to 0$ is exact, show it is a projective cover of M if and only if $Ker(f)$ is co-essential.

3. Find the Abelian groups with projective covers.

4. Prove that $Hom_R(M, M)$ local implies that M is indecomposable.

5. Prove that Q is an injective envelope of M if and only if M is a submodule of Q, and whenever M is a submodule of Q_i with Q_i injective, then thee exists a monomorphism $Q \to Q_i$ fixing M elementwise.

6. If R is left Noetherian and M is a left R-module, prove that M contains a maximal injective submodule.

7. If R is left Noetherian and left hereditary, and M is a left R-module, prove that M contains a unique maximal injective submodule which contains every injective submodule of M.

8. Prove that if A is essential in B and C is essential in D, then $A \oplus C$ is essential in $B \oplus D$.

9. If A is essential in B, and B is essential in C, prove that A is essential in C.

10. If A is essential in c, and B is essential in C, prove that $A \cap B$ is essential in C.

11. If A is a submodule of M, prove that M has a submodule B such that $A \cap B$ is essential in M.

12. Find an example of a module M with submodules S and N such that $S \subset N$, N is essential in M, but N/S is not essential in M/S.

13. Find an example of a module M with submodules A, B, and C such that A is essential in B, but such that $A + C$ is not essential in $B + C$.

14. Let $P = \{\sum_{i=0}^{\infty} a_i p^i : a_i \in Z\}$ for some prime p, and with $\sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{\infty} b_i p^i$ if p^{k+1} divides $\sum_{i=0}^{k} (a_i - b_i) p^i$ for all k. Add and multiply by the rules

$$\begin{aligned} \sum_{i=0}^{\infty} a_i p^i + \sum_{i=0}^{\infty} b_i p^i &= \sum_{i=0}^{\infty} (a_i + b_i) p^i \\ \left(\sum_{i=0}^{\infty} a_i p^i\right)\left(\sum_{i=0}^{\infty} b_i p^i\right) &= \sum_{i=0}^{\infty} (\sum_{j=0}^{i} a_{i-j} b_j) p^i \end{aligned}$$

Prove that P is a commutative integral domain. Prove that P is local with maximum ideal pP. Prove that $P \cong Hom_Z(Z(p^\infty), Z(p^\infty))$.

Chapter 4

Rings of Quotients

4.1 The Classical Left Quotient Ring

Definition 4.1.1 *An element of a ring R is* ***regular*** *if it is neither a left nor a right zero divisor.*

Definition 4.1.2 *A ring Q containing R is a* ***left quotient ring*** *of R if*

1. *every regular element in R has a two sided inverse in Q, and*
2. *every element in Q is of the form $a^{-1}b$, where a and b are in r, and a is regular in R.*

Definition 4.1.3 *(Ore's condition)*

Theorem 4.1.4 *(Ore, 1933) The ring R has a left quotient ring if and only if for a and b in R with b regular, there exist a_1 and b_1 in R with b_1 regular, such that $b_1 a = a_1 b$. (This condition is* ***Ore's condition****.) (Use usual proof)*

4.2 Goldie Rings

Definition 4.2.1 *Let S be a subset of R. The* ***left annihilator*** *of S is the set $l(S) = \{x \in R : xS = 0\}$. The* ***right annihilator*** *of S is $r(S) = \{x \in R : Sr = 0\}$. A left (right) ideal is a* ***left (right) annihilator*** *if it equals $l(S)$ $(r(S))$ for some subset S of R.*

Definition 4.2.2 *The ring R satisfies the* ***left Goldie chain conditions****, or R is a* ***left Goldie ring*** *if*

1. R satisfies the ascending chain condition on left annihilators, and
2. R contains no infinite direct sum of left ideals.

Theorem 4.2.3 *Let R be a left Goldie ring without zero divisors. Then R has a left quotient ring Q, and Q is a division ring.*

Definition 4.2.4 *The ring R is a **left Ore domain** if it has no zero divisors, and the intersection of any two non-zero left ideals of R is non-zero. (Ore domains are precisely those rings with a left ring of quotients which is a division ring.)*

Definition 4.2.5 *A ring is **prime** if $IJ = 0$ implies that $I = 0$ or $J = 0$ for ideals I and J. A ring is **semi-prime** if it has no non-zero nilpotent ideals.*

Remark 4.2.6 *A ring R is semi-prime if and only if R has no non-zero nilpotent left ideals if and only if R has no non-zero nilpotent right ideals.*

Lemma 4.2.7 *Let R be semi-prime with the ascending chain condition on left annihilators. If I and J are left ideals with $J \subset I$ and $r(I) \neq r(J)$, then there is an x in I such that $Ix \neq 0$ and $Ix \cap J = 0$.*

Theorem 4.2.8 *If R is semi-prime and has the ascending chain condition on left annihilators, then Rx and Ry essential imply Rxy essential.*

Theorem 4.2.9 *If R is semi-prime with the ascending chain condition on left annihilators, then Rx essential implies that x is regular.*

Lemma 4.2.10 *Let R be a semi-prime left Goldie ring. If x is in R and $l(x) = 0$, then Rx is essential and x is regular.*

Definition 4.2.11 *An ideal I is an **annihilator ideal** if I is the left annihilator of some left ideal.*

Lemma 4.2.12 *A non-zero minimal annihilator ideal of a semi-prime left Goldie ring R is a prime left Goldie ring (not necessarily with 1). Moreover, there is a finite direct sum of such ideals which is an essential left ideal of R.*

Lemma 4.2.13 *Let R be a semi-prime left Goldie ring. Then every essential left ideal of R contains a regular element.*

Theorem 4.2.14 *(Goldie) If R is a semi-prime left Goldie ring, then R has a left quotient ring.*

Lemma 4.2.15 *If Q is a left quotient ring of R, and $a_1, a_2, ..., a_n$ are regular elements of R, then there exist a, $b_1, b_2, ..., b_n$ in R with a regular, such that $a_i^{-1} = ab_i^{-1}$ for all i.*

Lemma 4.2.16 *If Q is a left quotient right of R, and $\{X_i\}_{i \in I}$ is an independent family of left ideals of R, then $\{QX_i\}_{i \in I}$ is an independent family of left ideals of Q.*

Theorem 4.2.17 *(Goldie) If R is a semi-prime left Goldie ring, its left ring of quotients is a semi-prime left Artinian ring.*

Theorem 4.2.18 *(Goldie) If R is a prime left Goldie ring, then R has a left ring of quotients $Q(R)$ which is a simple ring with the descending chain condition on left ideals. That is, $Q(R) = D_n$ for some division ring D.*

Definition 4.2.19 *Let R be a subring of S. Then R is a **left order** in S if S is a ring of left quotients for R.*

Theorem 4.2.20 *(Goldie) Let R be a left order in S, where S is semi-simple Artinian. Then R is a semi-prime left Goldie ring. If S is simple then R is prime.*

4.3 Noetherian Rings Satisfying The Regularity Conditions

Definition 4.3.1 *Let R be left Noetherian, N its ***maximum left nilpotent ideal***. The ring R satisfies the **regularity conditions** if an element a is regular in R if and only if $a + N$ is regular in R/N.*

Lemma 4.3.2 *Let R be left Noetherian and satisfy the regularity conditions. If an element a in R is regular, then for any x in $rad(R) \cap N$, $Ra : x$ contains a regular element.*

Lemma 4.3.3 *Let $T_0 = rad(N)$, $R_0 = R$, $R_{k+1} = R_k/T_k$, where $T_k = rad(N_k) \cap N_k$ with N_k the maximum nilpotent ideal of R_k for $k \geq 0$. For x in R, let $x_0 = x$ and $x_{k+1} = x_k + R_k$. Then if R satisfies the regularity conditions, and if a is regular in r, then a_k is regular in R_k.*

Lemma 4.3.4 *If R satisfies the regularity conditions, and if an element a is regular, then for any e in R with a_k in T_k, there exist f and f in R with g regular, such that $fa = ge$.*

Lemma 4.3.5 $N^s = 0$ *implies that* $N(R_i)^{s-1} = 0$.

Corollary 4.3.6 *Let R satisfy the regularity conditions, and suppose that the element a is regular. For x in R, there exist c and d in R with d regular, such that $ca = dx$.*

Theorem 4.3.7 *If R is left Noetherian an satisfies the regularity conditions, then R has a left quotient ring.*

Lemma 4.3.8 *Let R satisfy the regularity conditions. Then*

$$N(Q(R) = Q(R)(N(R)) = Q(R)(N(Q(R)) \cap R$$

*where $N(X)$ denotes the maximum nilpotent ideal of X. ***$Q(R)$ = injective hull?? ****

Lemma 4.3.9 *If R satisfies the regularity conditions, then $Q(R)/N(Q(R)) \cong Q(R/N)$.*

Theorem 4.3.10 *If R is left Noetherian and satisfies the regularity conditions, then R has a left quotient ring which is left Artinian.*

Theorem 4.3.11 *If R is left Noetherian and has a left quotient ring which is left Artinian, then R satisfies the regularity conditions.*

4.4 Serre Classes and Generalized Rings of Quotients

Definition 4.4.1 *A non-empty class $\mathcal{S}$ of left R-modules is a* ***Serre class*** *if it is closed under submodules, homomorphic images and extensions. It is a* ***strongly complete Serre class*** *if it is also closed under arbitrary direct sums.*

Examples

1. $R = Z$ and $\mathcal{S} =$ all p-groups
2. $R = Z$ and $\mathcal{S} =$ all bounded groups
3. For any ring R, $\mathcal{S} =$ all finite modules
4. For any ring R, $\mathcal{S} =$ all modules of finite length
5. For R Noetherian, $\mathcal{S} =$ all finitely generated groups

Definition 4.4.2 *If $\mathcal{S}$ is a Serre class, let $\mathcal{F}(\mathcal{S}) = \{I : I$ is a left ideal and $R/I \in \mathcal{S}\}$.*

Theorem 4.4.3 *$\mathcal{F}(\mathcal{S})$ is a filter of left ideals. If $\mathcal{S}$ is strongly complete, then so is $\mathcal{F}$.*

Definition 4.4.4 *Let $\mathcal{F}$ be a filter of left ideals, and $\mathcal{S}(\mathcal{F}) = \{M : (0 : m) \in \mathcal{F}$ for each $m \in M\}$.*

Theorem 4.4.5 *If $\mathcal{F}$ is a filter of left ideals, then $\mathcal{S}(\mathcal{F})$ is closed under submodules, homomorphic images and arbitrary direct sums, and so is a strongly complete additive class. If $\mathcal{F}$ is a strongly complete filter, then $\mathcal{S}(\mathcal{F})$ is a strongly complete Serre class.*

Theorem 4.4.6 *(Gabriel) If $\mathcal{F}$ is a strongly complete filter, then $\mathcal{F} = \mathcal{F}(\mathcal{S}(\mathcal{F}))$, and if $\mathcal{S}$ is a strongly complete Serre class, then $\mathcal{S} = \mathcal{S}(\mathcal{F}(\mathcal{S}))$.*

4.5 Exercises

1. $\mathcal{S}$ is a Serre class if and only if for every exact sequence $0 \to A \to B \to C \to 0$, $B \in \mathcal{S}$ if and only if A and C are in $\mathcal{S}$.

2. Let I be an ideal such that $I^2 = I$, and let $\mathcal{S}$ be the class of left modules M such that $IM = 0$. Then $\mathcal{S}$ is a Serre class.

3. Let $\overline{R}$ be the injective envelope of R. Prove that $\{I : Hom(R/I, \overline{R}) = 0$ is a strongly complete filter.

4. Let A be the set of all regular elements of R with the property that if $a \in A$, $b \in R$, there exist $a_1 \in A$, $b_1 \in R$ such that $a_1 b = b_1 a$. Let $\mathcal{F}_A = \{I : I$ is a left ideal, I contains an element of $A\}$. Prove that $\mathcal{F}_A$ is a filter of left ideals. Is it strongly complete?

Chapter 5

Self Injective Rings

5.1 Quasi-Frobenius Rings

Definition 5.1.1 *A ring R is **quasi-Frobenius** if R is left and right Artinian and $lr(I) = I$ and $rl(J) = j$ for left ideals I and right ideals J. (Note that one need only assume the ascending chain condition on either right or left ideals.)*

Examples. The following are QF (= quasi-Frobenius) rings, as will be seen in the sequel.

1. Semi-simple rings
2. The group ring $F(G)$ for any field and any finite group G
3. $R(G)$ for R QF and G finite
4. The ring of $n \cdot n$ matrices over a QF ring
5. A proper homomorphic image of a principal ideal domain
6. $F[T]$, where V is a finite dimensional vector space over a field F, and T a linear transformation on V

Remark 5.1.2 *In 6 above, what subspaces of V are injective? Is V injective? Can the fact that $F[T]$ is QF be used in studying linear transformations to any new extent. See exercise 408 in Curtis and Reiner.*

Remark 5.1.3 *Is R Noetherian if and only if R has at most $|R|$ left ideals? R Noetherian implies R has at most $|R|$ left ideals, clearly.*

Theorem 5.1.4 *The following are equivalent.*

1. *R is QF.*
2. *R is left Noetherian, $r(I_1 \cap I_2) = r(I_1) + r(I_2)$, and $rl(J) = J$ for left ideals I_i and right ideals J.*

3. *R is left and right Noetherian and left self injective.*

4. *R is left and right Noetherian and left self injective.*

5. *R is right Artinian and left self injective.*

6. *R is left Noetherian and left self injective.*

7. *R is right Noetherian and left self injective.*

8. *R satisfies the ascending chain condition on left annihilator ideals and is left self injective.*

Remark 5.1.5 *These are ring theoretical characterizations of QF rings. The proofs are in the equal. Following are lemmas, etc. needed in the proof.*

Lemma 5.1.6 *The following hold.*

1. If R is left self injective, then $r(I_1 \cap I_2) = r(I_1) + r(I_2)$ for left ideals I_i.

2. If R is left self injective, then $rl(J) = J$ for finitely generated right ideals J.

3. If $r(I_1 \cap I_2) = r(I_1) + r(I_2)$ and $rl(J) = J$ for finitely generated right ideals J, then every homomorphism from a finitely generated left ideal of R into R can be extended to R.

3. may not be quite correct

Lemma 5.1.7 *If R is right Artinian an $J_1 \subsetneqq J_2$ implies that $l(J_1) \supsetneqq l(j_2)$, for right ideals J_i, then for any non-zero left R-module M, $Hom_R(M, R) \neq 0$.*

Theorem 5.1.8 *If the radical N of R is nil, and $u_i + N$, $i = 1, 2, ..., k$ is a set of mutually orthogonal idempotents in R/N such that $\sum(u_i + N) = 1 + N$, then there exists a set e_i, $i = 1, 2, ..., k$ of mutually orthogonal idempotents such that $1 = \sum e_i$, and $e_i + N = u_i + N$. That is, idempotents can be lifted if the radical is nil.*

Lemma 5.1.9 *If R is left self injective, then $A(R) = \{r \in R : l(r)\ is\ left\ essential\} = N$, and R/N is (von Neumann) regular.*

Theorem 5.1.10 *The following categorical conditions are equivalent to the ring theoretical conditions stated earlier, that is, are equivalent to being QF.*

1. *$M \in_R \mathcal{M}$ is projective if and only if it is injective.*

2. *$M \in_R \mathcal{M}$ injective implies M is projective.*

3. *$M \in_R \mathcal{M}$ projective implies M in injective.*

Lemma 5.1.11 *(Faith) R satisfies the descending chain condition on left annihilator ideals if and only if to each left ideal I there corresponds a finitely generated left ideal $I_1 \subseteq I$ such that $r(I_1) = r(I)$.*

Lemma 5.1.12 *(Faith) If R is left self injective and satisfies the ascending chain condition on left annihilator ideals, and if $I \supset J$, I and J left ideals, then there exist finitely generated subideals I_1 and J_1 of I and J, respectively, such that $r(I_1) = r(I)$, $r(J_1) = r(J)$, and $I_1 \supset J_1$.*

Lemma 5.1.13 *If every injective in ${}_R\mathcal{M}$ is projective, then the injective envelope of a finitely generated module in ${}_R\mathcal{M}$ if finitely generated.*

Lemma 5.1.14 *(Chase) If R is semi-primary, then any $M \in_R \mathcal{M}$ satisfies the descending chain condition on finitely generated submodules.*

Lemma 5.1.15 *Suppose that N is nilpotent. Let p be a finitely generated projective in ${}_R\mathcal{M}$, $\Lambda = Hom_R(P, P)$, $Q = rad(\Lambda)$. then $Hom_R(P/NP, P/NP) \approx \Lambda/Q$.*

Lemma 5.1.16 *If R is left Artinian, Re is indecomposable, $e^2 = e$, then Re has a unique maximal submodule, namely Ne, where $N = rad(R)$. Furthermore, if Rf is indecomposable and $f^2 = f$, then $\mathrm{Re} \approx Rf$ if and only if $\mathrm{Re}/Ne \approx Rf/Nf$.*

Lemma 5.1.17 *A projective module P is a generator if every simple module is an image of P.*

Corollary 5.1.18 *If R is Artinian and P is an indecomposable projective, then P/NOP is simple.*